On the Stability of Hyperbolic Moment Equations

Julian Koellermeier

May 24th, 2016 Beijing Computational Science Research Center

Outline

Introduction to Moment Methods

Boltzmann Transport Equation

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

PDE for particles' *probability density function* $f(t, \mathbf{x}, \mathbf{c})$

- Describes change of f due to transport and collisions
- Collision operator S
- Usually a 7-dimensional phase space

Model Order Reduction

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

Ansatz in velocity space

$$f(t, \mathbf{x}, \mathbf{c}) = \sum_{i=0}^{M} f_i(t, \mathbf{x}) H_i \left(\frac{\mathbf{c} - \mathbf{v}}{\sqrt{\theta}} \right)$$

Model Order Reduction

$$\frac{\partial}{\partial t} f(t, \mathbf{x}, \mathbf{c}) + c_i \frac{\partial}{\partial x_i} f(t, \mathbf{x}, \mathbf{c}) = S(f)$$

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- shifted and scaled weighted Hermite polynomial H_i
- Galerkin approach leads to finite system of PDEs for coefficients f_i

A Short History of Hyperbolic Moment Equations

Grad's Method [Grad, 1949]

- Galerkin projection with Hermite polynomials, locally hyperbolic

Hyperbolic Moment Equations (HME) [Fan et al., 2012]

- modification of system matrix

Quadrature-Based Moment Equations (QBME) [JK, 2013]

- use of Gaussian quadrature

Operator Projection framework (OP) [Fan, JK et al., 2014]

- use of projections

Moment System

Boltzmann equation

$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0$$

$$\Downarrow$$

Hyperbolic Moment equations

$$\mathbf{D} \frac{\partial}{\partial t} \mathbf{w} + \mathbf{M}\mathbf{D} \frac{\partial}{\partial x} \mathbf{w} = \mathbf{0}$$

Moment System

Boltzmann equation

$$\frac{\partial}{\partial t} f + c \frac{\partial}{\partial x} f = 0$$

$$\Downarrow$$

Hyperbolic Moment equations

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\frac{\partial}{\partial x}\mathbf{w} = \mathbf{0}$$

Stability of Hyperbolic Moment Equations

Stability of Hyperbolic System

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{0}$$

Wave ansatz

$$\mathbf{u} = \mathbf{u}_0 \cdot e^{i(kx-\omega t)}, k \in \mathbb{R}, \omega \in \mathbb{C}, \quad Im(\omega) \leq 0 \text{ for stability.}$$

Stability Analysis

$$\begin{aligned} -i\omega\mathbf{u} + ik\mathbf{A}\mathbf{u} &= \mathbf{0} \\ (k\mathbf{A} - \omega\mathbf{I})\,\mathbf{u} &= \mathbf{0} \end{aligned}$$

$$\omega = EV(k\mathbf{A}) = k \cdot EV(\mathbf{A})$$

$\Rightarrow$ **A** needs to have only real eigenvalues.

Stability of Relaxation System

$$\partial_t \mathbf{u} = \varepsilon \mathbf{B} \mathbf{u}$$

Wave ansatz

$$\mathbf{u} = \mathbf{u}_0 \cdot e^{i(kx - \omega t)}, k \in \mathbb{R}, \omega \in \mathbb{C}, \quad Im(\omega) \leq 0 \text{ for stability.}$$

Stability Analysis

$$\begin{aligned} -i\omega \mathbf{u} &= \varepsilon \mathbf{B} \mathbf{u} \\ (i\varepsilon \mathbf{B} - \omega \mathbf{I})\, \mathbf{u} &= \mathbf{0} \end{aligned}$$

$$\omega = EV(i\varepsilon \mathbf{B}) = i\varepsilon \cdot EV(\mathbf{B})$$

$\Rightarrow$ **B** needs to have only negative (real) eigenvalues.

Stability of Hyperbolic Relaxation System

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \varepsilon \mathbf{B}\mathbf{u}$$

Wave ansatz

$$\mathbf{u} = \mathbf{u}_0 \cdot e^{i(kx-\omega t)}, k \in \mathbb{R}, \omega \in \mathbb{C}, \quad Im(\omega) \leq 0 \text{ for stability}$$

Stability analysis

$$\begin{aligned} -i\omega \mathbf{u} + ik\mathbf{A}\mathbf{u} &= \varepsilon \mathbf{B}\mathbf{u} \\ (k\mathbf{A} + i\varepsilon \mathbf{B} - \omega \mathbf{I})\,\mathbf{u} &= \mathbf{0} \end{aligned}$$

$$\omega = EV(k\mathbf{A} + i\varepsilon \mathbf{B})$$

$\Rightarrow k\mathbf{A} + i\varepsilon\mathbf{B}$ needs to have only eigenvalues with negative imaginary part

Instability of HME, [Zhao, Luo et al., 2015]

"Stability Analysis of a Globally Hyperbolic Moment System in One Dimension"

HME with BGK

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} \quad = \quad \varepsilon \mathbf{B}\mathbf{w}$$

Example: $n = 4$

Existence of eigenvalue with positive imaginary part and breakdown of numerical simulation.

Towards Stable Hyperbolic Moment Equations

HME with BGK

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} = \varepsilon \mathbf{B}\mathbf{w}$$

Stability analysis

$$\begin{aligned} \omega &= EV(k\mathbf{D}^{-1}\mathbf{M}\mathbf{D} + i\varepsilon\mathbf{B}) \\ &= EV(k\mathbf{M} + i\varepsilon\mathbf{D}\mathbf{B}\mathbf{D}^{-1}) \\ &= EV(k\mathbf{M} + i\varepsilon\mathbf{B}\mathbf{D}^{-1}) \end{aligned}$$

Towards Stable Hyperbolic Moment Equations

HME with BGK

$$\partial_t \mathbf{w} + \mathbf{D}^{-1}\mathbf{M}\mathbf{D}\partial_x \mathbf{w} = \varepsilon \mathbf{B}\mathbf{w}$$

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$$\begin{aligned} \omega &= EV(k\mathbf{D}^{-1}\mathbf{M}\mathbf{D} + i\varepsilon\mathbf{B}) \\ &= EV(k\mathbf{M} + i\varepsilon\mathbf{D}\mathbf{B}\mathbf{D}^{-1}) \\ &= EV(k\mathbf{M} + i\varepsilon\mathbf{B}\mathbf{D}^{-1}) \end{aligned}$$

Stable for $\mathbf{D} = diag(d_{ii})$!

SHME explanation

Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = 0$$

$$f = f_\alpha H_\alpha \Rightarrow \frac{\partial f}{\partial s} = \frac{\partial f_\alpha}{\partial s} H_\alpha + f_\alpha \frac{\partial H_\alpha}{\partial s}, \;\; s = t, x$$

Derivative relation for weighted Hermite polynomials

$$\frac{\partial H_\alpha}{\partial s} = \frac{\partial u}{\partial s} H_{\alpha+1} + \frac{1}{2} \frac{\partial \theta}{\partial s} H_{\alpha+2}$$

Recurrence relation for weighted Hermite polynomials

$$\xi H_\alpha = \theta H_{\alpha+1} + u H_\alpha + \alpha H_{\alpha-1}$$

Grad's Equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{Grad}} \partial_x \mathbf{w} = \frac{1}{\tau}\mathbf{B}\mathbf{w}$$

Grad model

$$\mathbf{A}_{\mathrm{Grad}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & \frac{3f_3}{2} & \theta & v \end{pmatrix}$$

Hyperbolic Moment Equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{HME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

HME model

$$\mathbf{A}_{\mathrm{HME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} & v & 4 \\ -\frac{f_3\theta}{\rho} & 0 & -f_3 & \theta & v \end{pmatrix}$$

Quadrature-Based Moment Equations

$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{QBME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

QBME model

$$\mathbf{A}_{\mathrm{QBME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 4f_3 & \frac{\rho\theta}{2} - \frac{10 f_4}{\theta} & v & 4 \\ -\frac{f_3\theta}{\rho} & 5f_4 & -f_3 & \theta + \frac{15 f_4}{\rho\theta} & v \end{pmatrix}$$

Stable Hyperbolic Moment Equations

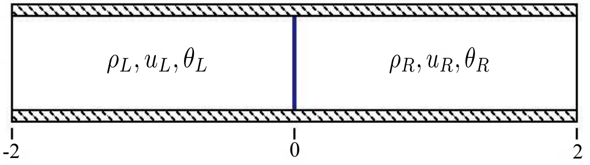
$$\partial_t \mathbf{w} + \mathbf{A}_{\mathrm{SHME}} \partial_x \mathbf{w} = \frac{1}{\tau} \mathbf{B} \mathbf{w}$$

SHME model

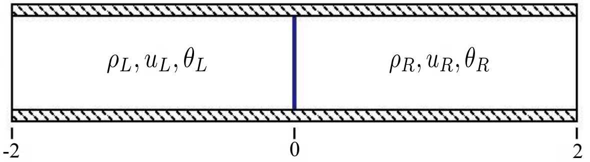
$$\mathbf{A}_{\mathrm{SHME}} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ \frac{\theta}{\rho} & v & 1 & 0 & 0 \\ 0 & 2\theta & v & \frac{6}{\rho} & 0 \\ 0 & 0 & \frac{\rho\theta}{2} & v & 4 \\ 0 & 0 & 0 & \theta & v \end{pmatrix}$$

Simulation Results

Shock Tube Test Case



Shock Tube Test Case



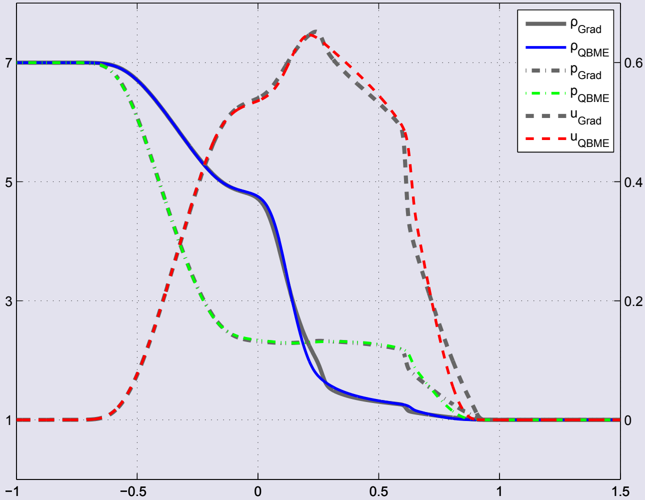
Riemann problem with BGK collision operator

$$\partial_t \mathbf{w} + \mathbf{A}(\mathbf{w})\partial_x \mathbf{w} = -\frac{1}{\tau}\mathbf{P}\mathbf{w}, \quad x \in [-2, 2]$$

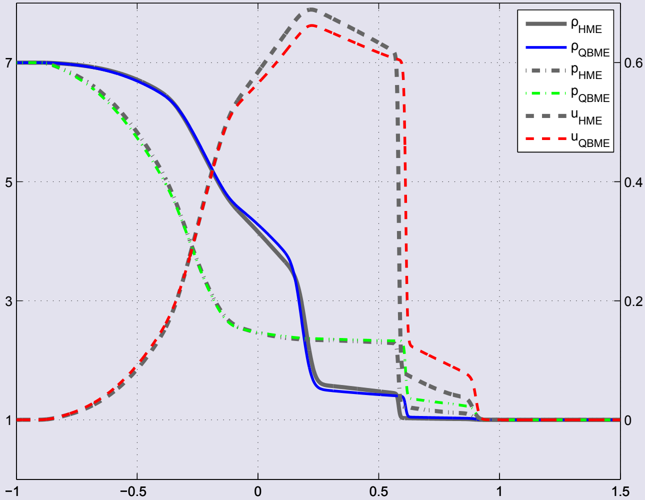
$$\rho_L = 7, \rho_R = 1$$

- Variable vector $\mathbf{w} = (\rho, u, \theta, f_3, f_4)$
- Relaxation time $\tau = \frac{\mathrm{Kn}}{\rho} \Rightarrow$ non-linear

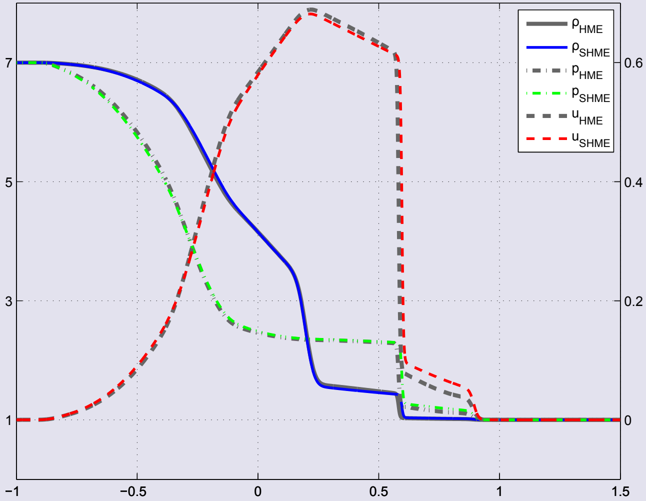
Grad vs QBME, $\mathrm{Kn} = 0.05$



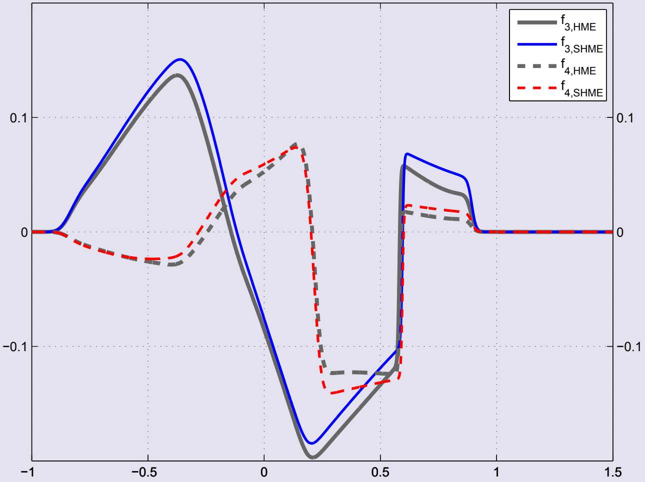
HME vs QBME, $\mathrm{Kn} = 0.5$



HME vs SHME, $\mathrm{Kn} = 0.5$



HME vs SHME, f_3, f_4



Summary and Further Work

Summary

(1) Stability of hyperbolic moment equations
(2) Instability of existing models
(3) New stable model: SHME
(4) Promising numerical results

Further Work

- Derivation of further stable moment models
- More simulations and test cases

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Thank you for your attention!