Miniwokshop:

High Order Reconstruction
and Well Balancing Techniques
for Hyperbolic
Conservation and Balance Laws

April 15-16, 2015

Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto 8/10, Torino, Italy

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Program

Wednesday 15th April (Aula 5, first floor)

13:30 - 14:00	Welcome
14:00 - 14:40	Gabriella Puppo
	Asymptotic preserving boundary conditions for kinetic models
14:45 - 15:25	Maurizio Tavelli
	A staggered semi-implicit arbitrary high order discontinuous Galerkin method for the Incompressible Navier-Stokes equations
15:30 - 16:00	Coffee Break
16:00 - 16:40	Birte Schmidtmann
	On the Relation between WENO3 and Third-Order Limiter Functions in Finite Volume Methods
16:45 - 17:25	Mathea J. Vuik
	Automated parameters for troubled-cell indicators using outlier detection

Thursday 16th April (Aula C, courtyard)

09:00 - 09:40	Pep Mulet - *Accuracy analysis of finite difference Weighted Essentially Non Oscillatory schemes and boundary extrapolation techniques for complex domains*
09:45 - 10:25	Matteo Semplice - *A third order h-adaptive finite volume solver based on CWENO and the numerical entropy production*
10:30 - 11:00	Coffee Break
11:00 - 11:40	Jennifer Ryan - *Smoothness-Increasing Accuracy-Conserving filtering for discontinuous Galerkin methods with source terms*
11:45 - 12:25	Valerio Caleffi - *A comparison between well-balanced numerical approaches for the simulation of shallow flows on bottom discontinuities*
12:30 - 13:10	Michael Dumbser - *A posteriori subcell limiting of the discontinuous Galerkin finite element method for hyperbolic conservation laws*
Around 13:15	Closing

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Asymptotic preserving boundary conditions for kinetic models

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In this work we propose asymptotic preserving boundary conditions for kinetic problems. We concentrate on the BGK model, and illustrate with several examples why and how the asymptotic limit must be preserved also when imposing boundary conditions. Next, we illustrate the robustness of the method with examples involving rarefied flow at several regimes on non trivial geometries.

Recently there has been great attention to the study of Asymptotic Preserving (AP) schemes (see the recent review [2]), but the same care has not been bestowed on the enforcement of boundary conditions. However, numerical experiments show clearly that, if proper boundary conditions are not imposed, spurious effects are to be expected, which prevent convergence to the correct asymptotic limit, [3].

We start from illustrating the problem, and proposing a solution. Next we discuss how to implement kinetic BGK and ES-BGK models preserving the correct asymptotic limits.

The talk will also illustrate applications. In particular, we will consider the passive transport of a set of particles on a rarefied flow in a nozzle. These simulations provide useful data for the study of the behaviour of pollutants, composed of unburnt particles, ejected from satellite thrusters.

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A staggered semi-implicit arbitrary high order discontinuous Galerkin method for the Incompressible Navier-Stokes equations

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We present a new arbitrary high order accurate semi-implicit discontinuous Galerkin (DG) method for the solution of the incompressible Navier-Stokes equations on staggered unstructured curved meshes.

The scheme is based on the general ideas proposed in [1, 2] for the arbitrary high order in space and it is then extended to a fully space-time high order in [3] for the two dimensional case.

While the discrete pressure is defined on the primal grid, the discrete velocity field is defined on an edge-based dual grid. The method is designed in such a fashion that the entire velocity vector is defined on the edge-based dual control volumes. Formal substitution of the discrete momentum equation into the discrete continuity equation yields a sparse block four-diagonal linear equation system for the scalar pressure. The high order in time is then achieved by considering a pressure gradient formulation on space-time control volumes and introducing a Picard iteration to update the nonlinear convection.

The flexibility of high order DG methods on curved unstructured meshes allows to discretize even complex physical domains with rather coarse grids. For the space high order case the main system for the pressure results symmetric and at least semi-positive definite and hence it allows to use fast iterative linear solver such as CG method. In addition, all the volume and surface integrals needed by the scheme depend only on the geometry and the polynomial degree of the basis and test functions and can therefore be precomputed and stored in a preprocessing stage. This leads to a significant saving in terms of computational effort for the time evolution part. In this way also the extension to a fully curved isoparametric approach becomes natural and affects only the preprocessing step.

The method can then be extended to a fully three-dimensional arbitrary high order scheme based on staggered tetrahedral-hexahedral elements. Also in this case, the resulting main system for the pressure is symmetric and in general semi-positive definite up to the first order in time, extendable to the second order using a Crank-Nicolson procedure. On the contrary, we loose the symmetry but we get arbitrary high order results also in time, crucial to obtain good solutions even when high unsteady problems or huge time steps have been considered.

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On the Relation between WENO3 and Third-Order Limiter Functions in Finite Volume Methods

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We are interested in the numerical solution of hyperbolic conservation laws on the most local compact stencil consisting of only nearest neighbors. In the Finite Volume setting, the main challenge is the reconstruction of the interface values which are crucial for the definition of the numerical flux functions and thus, for the order of accuracy of the resulting scheme. Often, the functions of interest contain smooth parts as well as large gradients, discontinuities, or shocks. Treating such functions with high-order schemes may lead to undesired effects such as oscillations. However, what is required is a solution with sharp discontinuities while maintaining high-order accuracy in smooth regions. One possible way of achieving this is the use of limiter functions in the MUSCL framework which switch the reconstruction to lower order when necessary. Another possibility is the third-order variant of the WENO family, called WENO3 which was introduced by Jiang and Shu [3]. In this work, we will recast both methods in the same framework to demonstrate the relation between Finite Volume limiter functions and the way WENO3 performs limiting. Special attention is given to the limiter function developed by Čada and Torrilhon [2], which is based on the local double logarithmic reconstruction function of Artebrant and Schroll [1]. The limiter contains a decision criterion which distinguishes between discontinuities and smooth extrema, containing a parameter r, which is the radius of a so-called asymptotic region, [2]. Unfortunately, r remains unspecified. Our analysis shows that r can be coupled to the solution. Thus, our newly-developed limiter function does not require an artificial parameter. Instead, it uses only information of the initial condition. We compare our insights with the formulation of the weight-functions in WENO3. The weights as defined in [3] contain a parameter ε which was originally introduced to avoid the division by zero. However, we will show that ε has a significant influence on the behavior of the reconstruction and relating ε to r allows us to give a clarifying interpretation. The comparison of both approaches proves fruitful for either one and allows further insight into limiter functions.

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Automated parameters for troubled-cell indicators using outlier detection

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In general, solutions of nonlinear hyperbolic PDE's contain shocks or develop discontinuities. One option for improving the numerical treatment of the spurious oscillations that occur near these artifacts is through the application of a limiter. The cells where such treatment is necessary are referred to as troubled cells, which are selected using a troubled-cell indicator. Examples are the KXRCF shock detector, the minmod-based TVB indicator, and the modified multiwavelet troubled-cell indicator.

The current troubled-cell indicators perform well as long as a suitable, problem-dependent parameter is chosen. An inappropriate choice of the parameter will result in detection of too few or too many elements. Detection of too few elements leads to spurious oscillations, since not enough elements are limited. If too many elements are detected, then the limiter is applied too often, and therefore the method is more costly and the approximation smooths out after a long time. The optimal parameter is chosen such that the minimal number of troubled cells is detected and the resulting approximation is free of spurious oscillations. In general, many tests are required to obtain this optimal parameter for each problem.

In this presentation, we will see that the sudden increase or decrease of the indicator value with respect to the neighboring values is important for detection. Indication basically reduces to detecting the outliers of a vector (one dimension) or matrix (two dimensions). This is done using Tukey's boxplot approach to detect which coefficients in a vector are straying far beyond others [2].

We provide an algorithm that can be applied to various troubled-cell indication variables. Using this technique, the problem-dependent parameter that the original indicator requires, is no longer necessary, as the parameter will be chosen automatically.

We will apply this technique to the modified multiwavelet troubled-cell indicator [3, 4], which can be used to detect discontinuities in (the derivatives of) the DG approximation. Here, Alpert's multiwavelet basis is used [1]. We will use either the original indicator (with an optimal parameter), or the outlier-detection technique. In that way, the performance of the new technique can be easily compared to the current method.

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Accuracy analysis of finite difference Weighted Essentially Non Oscillatory schemes and boundary extrapolation techniques for complex domains

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Finite difference WENO schemes [4, 5] have become an efficient method for the approximate solution of multidimensional hyperbolic conservation laws. These schemes follow a method of lines strategy, for which the spatial discretization is obtained by numerical differentiation of reconstruction of fluxes. Higher order accuracy can then be obtained from highly accurate reconstructions.

In [1] we analyze Jiang-Shu's smoothness indicators [4], for any stencil length, and their derived weights, obtaining that they have accuracy properties that give maximal order schemes. In [2] we analyze the faster converging weights proposed in [8] to conclude that the accuracy of the schemes obtained from these weights near discontinuities deteriorate with respect to classical ENO schemes. We obtain a modification of these weights with both faster converging properties at smooth regions, regardless of neighboring extrema, and enough accuracy near discontinuities.

Finite difference WENO schemes can be quite readily designed as long as the underlying mesh is an equispaced Cartesian mesh. In this context, the application of suitable numerical boundary conditions for hyperbolic conservation laws on domains with complex geometry has become a problem with certain difficulty that has been tackled in different ways according to the nature of the numerical methods and mesh type ([6, 7]). In [3] we propose an extrapolation technique on structured Cartesian meshes (which, as opposed to non-structured meshes, can not be adapted to the morphology of the domain boundary) of the information in the interior of the computational domain to ghost cells. This technique is based on the application of Lagrange interpolation with a previous filter for the detection of discontinuities that permits a data dependent extrapolation, with higher order at smooth regions and essentially non oscillatory properties near discontinuities.

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A third order h-adaptive finite volume solver based on CWENO and the numerical entropy production

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This work is part of a project to construct high order h-adaptive finite volume scheme. Recently the idea of numerical entropy production [1] was extended to the case of balance laws [2] and this suggest to look for third order reconstruction procedure that can maintain its accuracy on non-uniform meshes.

In this respect, the Compact-WENO (CWENO) reconstruction of [3] is a more viable candidate than the traditional WENO reconstruction of third order. In fact, this latter relies on coefficients that depend on the mesh geometry and that would need to be recomputed after each mesh adaption step. Moreover, CWENO computes a reconstruction polynomial that is uniformly accurate in the whole cell and thus can be employed to compute a third order accurate reconstruction at interior points of the cell, which is needed for well-balanced quadratures. In this work, we extend the CWENO reconstruction to non-uniform meshes in one and two space dimensions and employ it in the construction of third order accurate h-adaptive schemes for conservation [2] and balance laws[2]. A short discussion on the role of the parameter appearing in the nonlinear weights will be provided [5]. Numerous tests compare the WENO and CWENO reconstructions and illustrate the behaviour of the resulting scheme.

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Smoothness-Increasing Accuracy-Conserving filtering for discontinuous Galerkin methods with source terms

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Discontinuous Galerkin (DG) methods exhibit hidden accuracy which should be used to further the betterment of the approximation. One method is to implement a convolution kernel approach that improves the order of accuracy from $k + 1$ to order $2k + m$ for time-dependent linear convection-diffusion equations, where k is the highest degree polynomial used in the approximation and m depends upon the choice of the numerical flux.

In this talk, we discuss the accuracy enhancing capabilities of the Smoothness-Increasing Accuracy-Conserving (SIAC) filter for DG solutions to hyperbolic equations with smooth source terms. We discuss theoretical and computational results and the possibilities for accuracy enhancement. We implement the Smoothness-Increasing Accuracy-Conserving filter that enhances accuracy by utilizing information contained in the negative-order norm. The local postprocessing technique that makes use of the information contained in the negative-order norm was originally developed by Bramble and Schatz in the context of continuous finite element methods for elliptic problems [1]. They demonstrated that it is possible to construct a better approximation by convoluting the finite element solution with a local averaging operator in the neighborhood of a point x, where the convergence in the negative-order norms was higher than L^2-norm. Cockburn, Luskin, Shu, and Suli established a framework to apply this technique to linear hyperbolic equations in the context of the Discontinuous Galerkin methods [2]. Numerical experiments showed that the post-processing had a positive impact on nonlinear hyperbolic equations [3, 4]. This technique is labelled as a SIAC filter and was extended to nonuniform meshes, higher-order derivatives, and as a filtering technique to improve the visualization of streamlines [5, 6, 7].

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A comparison between well-balanced numerical approaches for the simulation of shallow flows on bottom discontinuities

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In last two decades, in the context of the Shallow Water Equations (SWE) numerical integration, the efforts of many researchers were devoted to conceive techniques for the exact preservation of motionless state over non-flat bottom. Recently, such efforts are mainly oriented to the proper treatment of the bottom discontinuities and to the exact preservation of moving water steady flow. The well-known definition of well-balanced scheme for quiescent flow is therefore extended to steady flows. At the present state of art a large number of techniques for the well-balancing of a SWE model in the case of a quiescent flow exists whilst very few approaches for the well-balancing of a moving steady flow are available.

In this work we give a contribution to the well-balancing of SWE models for moving steady flow, indicating how some key elements of the standard approaches have to be changed to improve the overall behavior of the schemes. We have focused our attention on both the classical finite volume formulation and the path-conservative formulation, inside a unified context represented by third-order accurate discontinuous Galerkin schemes. In particular, a comparison between five numerical treatments of the bottom discontinuities is presented.

We consider three widespread approaches that perform well if the motionless state has to be preserved. First, a simple technique, which consists in a proper initialization of the bed elevation that imposes the continuity of the bottom profile is taken into account [1]. Then, we consider the hydrostatic reconstruction method [2] and a path-conservative scheme based on a linear integration path [3].

We than consider two further approaches (the former characterized by a limited diffusion and the latter original) which are promising for the preservation of a moving-water steady state. The former is obtained modifying the hydrostatic reconstruction as suggested in [4]. This method is characterized by a correction of the numerical flux based on the local conservation of the total head. The last model is obtained improving the path-conservative scheme using a curvilinear path. The non-linear path is defined imposing the local conservation of the total head and the discharge at the cell-interfaces.

Several test cases are used to verify the accuracy, the well-balancing, the behavior in simulating a quiescent flow and the resolution of the models in simulating unsteady flows. A specific test case is also introduced to highlight the difference between the five schemes when a steady moving flow interacts with a bottom step.

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A posteriori subcell limiting of the discontinuous Galerkin finite element method for hyperbolic conservation laws

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We propose a novel a posteriori finite volume subcell limiter technique for the Discontinuous Galerkin finite element method for nonlinear systems of hyperbolic conservation laws in multiple space dimensions that works well for arbitrary high order of accuracy in space and time and that does not destroy the natural subcell resolution properties of the DG method. High order time discretization is achieved via a one-step ADER approach that uses a local space-time discontinuous Galerkin predictor method to evolve the data locally in time within each cell.

Our new limiting strategy is based on the so-called MOOD paradigm, which a posteriori verifies the validity of a discrete candidate solution against physical and numerical detection criteria after each time step. Here, we employ a relaxed discrete maximum principle in the sense of piecewise polynomials and the positivity of the numerical solution as detection criteria. Within the DG scheme on the main grid, the discrete solution is represented by piecewise polynomials of degree N. For those troubled cells that need limiting, our new limiter approach recomputes the discrete solution by scattering the DG polynomials at the previous time step onto a set of $N_s = 2N + 1$ finite volume subcells per space dimension. A robust but accurate ADER-WENO finite volume scheme then updates the subcell averages of the conservative variables within the detected troubled cells. The recomputed subcell averages are subsequently gathered back into high order cell-centered DG polynomials on the main grid via a subgrid reconstruction operator. The choice of $N_s = 2N + 1$ subcells is optimal since it allows to match the maximum admissible time step of the finite volume scheme on the subgrid with the maximum admissible time step of the DG scheme on the main grid, minimizing at the same time also the local truncation error of the subcell finite volume scheme. It furthermore provides an excellent subcell resolution of discontinuities.

Our new approach is therefore radically different from classical DG limiters, where the limiter is using TVB or (H)WENO reconstruction based on the discrete solution of the DG scheme on the main grid at the new time level. In our case, the discrete solution is recomputed within the troubled cells from the old time level using a different and more robust numerical scheme on a subgrid level.

We illustrate the performance of the new aposteriori subcell ADER-WENO finite volume limiter approach for very high order DG methods via the simulation of numerous test cases run on Cartesian grids in two and three space dimensions, using DG schemes of up to *tenth* order of accuracy in space and time $N = 9$. The method is also able to run on massively parallel large scale supercomputing infrastructure, which is shown via one 3D test problem that uses 10 billion space-time degrees of freedom per time step.

We will furthermore show extensions of the present approach to space-time adaptive Cartesian grids (AMR).

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